

Covariant perturbation expansion of off-diagonal heat kernel

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ABSTRACT: Covariant perturbation expansion is an important method in quantum field theory. In this paper an expansion up to arbitrary order for off-diagonal heat kernels in flat space based on the covariant perturbation expansion is given. In literature, only diagonal heat kernels are calculated based on the covariant perturbation expansion.

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Contents

1	Introduction	1
2	The covariant perturbation expansion	2
2.1	The heat kernel	2
2.2	The covariant perturbation expansion	2
3	The off-diagonal heat kernel	4
4	An alternative expression of the n-th order heat kernels	6
5	The first three orders of diagonal and global heat kernels	7
5.1	The heat kernel with respect to $\tilde{V}(k_n)$	7
5.2	The heat kernel with respect to $V(R_n)$	9
5.3	The heat kernel with respect to u_n	11
6	Conclusion	12

1 Introduction

The heat-kernel method is widely used in physics [1, 2] and mathematics [1, 3]. The heat-kernel method can extract the information embedded in an operator; in physics the operator is often the Hamiltonian [1, 2]. In quantum field theory, for example, one can calculate one-loop effective actions, vacuum energies, and other spectral functions by the heat-kernel method [1, 4, 5]. In statistical mechanics, the heat-kernel method is very useful since the global heat kernel, the trace of the heat kernel, is indeed the partition function [6–8]. In scattering theory, the heat-kernel method can be used to calculate scattering phase shifts [9, 10].

Developing various asymptotic expansions is a core issue in the study of the heat-kernel method. There is much research on the perturbation calculation of heat kernels. Nevertheless, most literature dealing with only diagonal heat kernel [1, 11–13], while there are only a very few literature calculates off-diagonal heat kernel [14–18]. Nevertheless, rather than the diagonal one, the off-diagonal heat kernel contains all information of an operator, and many problems need off-diagonal, e.g., the heat-kernel method for scattering [9, 10].

The covariant perturbation expansion, among various heat-kernel algorithms, is a very effective method [19–29]. It can account for non-local effects [30]. Particularly, the covariant perturbation expansion is uniformly convergent [20, 27]. The special advantage of the covariant perturbation, as an example, is shown in Ref. [9]. It is provided in Ref. [9] a heat-kernel method to calculate scattering phase shifts. By this method, each method of

calculating heat kernels can be converted into a method of solving a scattering problem. It is shown that the perturbation expansion of phase shift obtained by the covariant perturbation expansion is more accurate than the Born approximation.

In a series of papers, only the diagonal heat kernel is considered [5]. In this paper, we expand the off-diagonal heat kernel to arbitrary order for flat space cases by the covariant perturbation theory.

In section 2, we provide an expansion of heat kernels based on the covariant perturbation expansion. In section 3, we give an expansion of off-diagonal heat kernels. In section 4, we give an alternative expression of off-diagonal heat kernels. In section 5, the first three orders of diagonal and global heat kernels are given. The conclusion is given in section 6.

2 The covariant perturbation expansion

2.1 The heat kernel

For an operator D , we can define the heat kernel operator [1],

$$K(\tau) = e^{-D\tau}. \quad (2.1)$$

The heat kernel operator is determined by the heat equation

$$\frac{d}{d\tau} K(\tau) + DK(\tau) = 0 \quad (2.2)$$

with the initial condition $K(0) = 1$.

The heat kernel is the matrix element of the heat kernel operator [5]

$$K(\tau; x, y) = \langle x | K(\tau) | y \rangle = \sum_{\lambda} e^{-\lambda\tau} \phi_{\lambda}(x) \phi_{\lambda}^*(y) \quad (2.3)$$

and the global heat kernel is the trace of the heat kernel

$$K_D(\tau) = \text{tr} K(\tau) = \int dx K(\tau; x, x) = \sum_{\lambda} e^{-\lambda\tau}, \quad (2.4)$$

where $\phi_{\lambda}(x)$ and λ are the eigenfunction and eigenvalue of the operator D , determined by the eigenequation $D\phi_{\lambda} = \lambda\phi_{\lambda}$.

2.2 The covariant perturbation expansion

In Refs. [5, 20, 25, 26], the covariant perturbation expansion is applied to calculate diagonal heat kernels to first three orders. In Ref. [9], for calculating the scattering phase shift, we use the covariant perturbation expansion to calculate the non-diagonal heat kernel. In this paper, we give an expansion of non-diagonal heat kernels to n -th order by the covariant perturbation method.

The expansion of a heat kernel operator in flat space can be expressed as [5]

$$K_D(\tau) = K_0(\tau) + K_1(\tau) + K_2(\tau) + \cdots, \quad (2.5)$$

where $K_n(\tau)$ is the n -th order heat kernel operator. In our cases,

$$D = -\nabla^2 + V. \quad (2.6)$$

In order to achieve the non-diagonal heat kernel, from Eqs. (2.2) and (2.5), we achieve the following recurrence relations:

$$\begin{cases} \frac{dK_0}{d\tau} = \nabla^2 K_0, & K_0(0) = 1 \\ \frac{dK_n}{d\tau} = \nabla^2 K_n - V K_{n-1}, & K_n(0) = 0 \end{cases}. \quad (2.7)$$

The solution of Eq. (2.7) can be expressed formally as

$$K_0(\tau) = \exp(\tau \nabla^2), \quad (2.8)$$

$$K_n(\tau) = K_0(\tau) C_n(\tau). \quad (2.9)$$

Note that $C_n(\tau)$ here is an operator.

Derivating Eq. (2.8) with respect to τ gives

$$\frac{d}{d\tau} K_n(\tau) = \nabla^2 K_0 C_n(\tau) + K_0(\tau) \frac{d}{d\tau} C_n(\tau). \quad (2.10)$$

From Eqs. (2.7) and (2.10), we obtain

$$K_0(\tau) \frac{d}{d\tau} C_n(\tau) = -V K_{n-1}(\tau). \quad (2.11)$$

By integrating both sides of Eq. (2.11) with respect to τ , we obtain

$$C_n(\tau) = \int_0^\tau d\tau_n K_0^{-1}(\tau_n) (-V) K_{n-1}(\tau_n), \quad (2.12)$$

where $C_n(0) = 0$.

By Eq. (2.8), we obtain

$$K_0(\tau) K_0(\tau_1) = K_0(\tau + \tau_1), \quad \tau > 0, \quad \tau_1 > 0, \quad (2.13)$$

$$K_0^{-1}(\tau) = K_0(-\tau). \quad (2.14)$$

By Eqs. (2.12), (2.13), and (2.14), we arrive at

$$K_n(\tau) = \int_0^\tau d\tau_n K_0(\tau - \tau_n) (-V) K_{n-1}(\tau_n). \quad (2.15)$$

Repeatedly using Eq. (2.15), we obtain

$$\begin{aligned} K_n(\tau) &= \int_0^\tau d\tau_n \int_0^{\tau_n} d\tau_{n-1} \cdots \int_0^{\tau_3} d\tau_2 \int_0^{\tau_2} d\tau_1 \\ &\quad \times K_0(\tau - \tau_n) (-V) K_0(\tau_n - \tau_{n-1}) (-V) \cdots (-V) K_0(\tau_2 - \tau_1) (-V) K_0(\tau_1). \end{aligned} \quad (2.16)$$

3 The off-diagonal heat kernel

Before we calculate the n -th order matrix elements of heat kernels, we first need to calculate the matrix elements of the leading order heat kernel.

The matrix element of a leading order heat kernel in the coordinate representation reads [5]

$$\langle x | K_0(\tau) | y \rangle = \frac{1}{(4\pi\tau)^{\nu/2}} e^{-(x-y)^2/(4\tau)}, \quad (3.1)$$

where ν denotes the dimension of space.

Now we calculate the n -th order matrix elements of a heat kernel $\langle x | K_n | y \rangle$ based on Eq. (2.16).

Inserting the identity $1 = \int d^\nu z |z\rangle \langle z|$ into $\langle x | K_n | y \rangle$, we have

$$\begin{aligned} \langle x | K_n | y \rangle &= (-1)^n \int_0^\tau d\tau_n \int_0^{\tau_n} d\tau_{n-1} \cdots \int_0^{\tau_3} d\tau_2 \int_0^{\tau_2} d\tau_1 \int d^\nu z_{2n} \int d^\nu z_{2n-1} \cdots \int d^\nu z_2 \int d^\nu z_1 \\ &\quad \times \langle x | K_0(\tau - \tau_n) | z_{2n} \rangle \langle z_{2n} | V | z_{2n-1} \rangle \langle z_{2n-1} | K_0(\tau_n - \tau_{n-1}) | z_{2n-2} \rangle \cdots \langle z_2 | V | z_1 \rangle \langle z_1 | K_0(\tau_1) | y \rangle. \end{aligned} \quad (3.2)$$

Substituting $\langle x | V | y \rangle = V(x) \delta(x - y)$ into Eq. (3.2) gives

$$\begin{aligned} \langle x | K_n | y \rangle &= (-1)^n \int_0^\tau d\tau_n \int_0^{\tau_n} d\tau_{n-1} \cdots \int_0^{\tau_3} d\tau_2 \int_0^{\tau_2} d\tau_1 \int d^\nu z_{2n} \int d^\nu z_{2n-2} \cdots \int d^\nu z_2 \\ &\quad \times \langle x | K_0(\tau - \tau_n) | z_{2n} \rangle V(z_{2n}) \langle z_{2n} | K_0(\tau_n - \tau_{n-1}) | z_{2n-2} \rangle V(z_{2n-2}) \cdots V(z_2) \langle z_2 | K_0(\tau_1) | y \rangle. \end{aligned} \quad (3.3)$$

By Eq. (3.1), we arrive at

$$\begin{aligned} \langle x | K_n | y \rangle &= (-1)^n \int_0^\tau d\tau_n \int_0^{\tau_n} d\tau_{n-1} \cdots \int_0^{\tau_3} d\tau_2 \int_0^{\tau_2} d\tau_1 \int d^\nu z_{2n} \int d^\nu z_{2n-2} \cdots \int d^\nu z_2 \\ &\quad \times \frac{\exp\left(-\frac{(x-z_{2n})^2}{4(\tau-\tau_n)}\right)}{[4\pi(\tau-\tau_n)]^{\nu/2}} V(z_{2n}) \frac{\exp\left(-\frac{(z_{2n}-z_{2n-2})^2}{4(\tau_n-\tau_{n-1})}\right)}{[4\pi(\tau_n-\tau_{n-1})]^{\nu/2}} V(z_{2n-2}) \cdots V(z_2) \frac{\exp\left(-\frac{(z_2-y)^2}{4\tau_1}\right)}{(4\pi\tau_1)^{\nu/2}}. \end{aligned} \quad (3.4)$$

The Fourier transform of $V(z)$ is

$$V(z) = \int \frac{d^\nu k}{(2\pi)^{\nu/2}} e^{ikz} \tilde{V}(k). \quad (3.5)$$

Substituting Eq. (3.5) into Eq. (3.4), we obtain

$$\begin{aligned} \langle x | K_n | y \rangle &= (-1)^n \int_0^\tau d\tau_n \int_0^{\tau_n} d\tau_{n-1} \cdots \int_0^{\tau_3} d\tau_2 \int_0^{\tau_2} d\tau_1 \\ &\quad \times \int \frac{d^\nu k_n}{(2\pi)^{\nu/2}} \tilde{V}(k_n) \int \frac{d^\nu k_{n-1}}{(2\pi)^{\nu/2}} \tilde{V}(k_{n-1}) \cdots \int \frac{d^\nu k_1}{(2\pi)^{\nu/2}} \tilde{V}(k_1) I_1, \end{aligned} \quad (3.6)$$

where

$$I_1 = \int d^\nu z_{2n} \int d^\nu z_{2n-2} \cdots \int d^\nu z_2 \frac{\exp \left(-\frac{(x-z_{2n})^2}{4(\tau-\tau_n)} - \frac{(z_{2n}-z_{2n-2})^2}{4(\tau_n-\tau_{n-1})} - \cdots - \frac{(z_2-y)^2}{4\tau_1} + ik_n z_{2n} + ik_n z_{2n-2} + \cdots + ik_1 z_2 \right)}{[4\pi(\tau-\tau_n)]^{\nu/2} [4\pi(\tau_n-\tau_{n-1})]^{\nu/2} \cdots (4\pi\tau_1)^{\nu/2}}. \quad (3.7)$$

The integral in I_1 can be performed by using the Gaussian integral [5],

$$\begin{aligned} & \int d^\nu x \exp \left(-A|x-a|^2 - B|x-b|^2 + 2cx \right) \\ &= \frac{\pi^{\nu/2}}{(A+B)^{\nu/2}} \exp \left(-\frac{AB|a-b|^2}{A+B} + \frac{2c(Aa+Bb)+|c|^2}{A+B} \right). \end{aligned} \quad (3.8)$$

Then we arrive at

$$\begin{aligned} \langle x | K_n | y \rangle &= (-1)^n \frac{e^{-(x-y)^2/(4\tau)}}{(4\pi\tau)^{\nu/2}} \int_0^\tau d\tau_n \cdots \int_0^{\tau_3} d\tau_2 \int_0^{\tau_2} d\tau_1 \int \frac{d^\nu k_n}{(2\pi)^{\nu/2}} \tilde{V}(k_n) \cdots \int \frac{d^\nu k_1}{(2\pi)^{\nu/2}} \tilde{V}(k_1) \\ &\times \exp \left(ik_1 \left(\frac{\tau_1}{\tau} x + \frac{\tau-\tau_1}{\tau} y \right) + ik_2 \left(\frac{\tau_2}{\tau} x + \frac{\tau-\tau_2}{\tau} y \right) + \cdots \right. \\ &+ ik_{n-1} \left(\frac{\tau_{n-1}}{\tau} x + \frac{\tau-\tau_{n-1}}{\tau} y \right) + ik_n \left(\frac{\tau_n}{\tau} x + \frac{\tau-\tau_n}{\tau} y \right) \\ &- \left[\frac{(\tau-\tau_1)\tau_1}{\tau} k_1^2 + \frac{(\tau-\tau_2)\tau_2}{\tau} k_2^2 + \cdots + \frac{(\tau-\tau_{n-1})\tau_{n-1}}{\tau} k_{n-1}^2 + \frac{(\tau-\tau_n)\tau_n}{\tau} k_n^2 \right] \\ &- 2 \left[\frac{(\tau-\tau_2)\tau_1}{\tau} k_1 k_2 + \cdots + \frac{(\tau-\tau_{n-1})\tau_1}{\tau} k_1 k_{n-1} + \frac{(\tau-\tau_n)\tau_1}{\tau} k_1 k_n \right] \\ &- 2 \left[\frac{(\tau-\tau_3)\tau_2}{\tau} k_2 k_3 + \cdots + \frac{(\tau-\tau_{n-1})\tau_2}{\tau} k_2 k_{n-1} + \frac{(\tau-\tau_n)\tau_2}{\tau} k_2 k_n \right] - \cdots \\ &- 2 \left[\frac{(\tau-\tau_{n-1})\tau_{n-2}}{\tau} k_{n-2} k_{n-1} + \frac{(\tau-\tau_n)\tau_{n-2}}{\tau} k_{n-2} k_n \right] - 2 \frac{(\tau-\tau_n)\tau_{n-1}}{\tau} k_{n-1} k_n \Big). \end{aligned} \quad (3.9)$$

Introducing

$$R_1 = \frac{\tau_1}{\tau} x + \frac{\tau-\tau_1}{\tau} y, R_2 = \frac{\tau_2}{\tau} x + \frac{\tau-\tau_2}{\tau} y, \cdots, R_n = \frac{\tau_n}{\tau} x + \frac{\tau-\tau_n}{\tau} y \quad (3.10)$$

and using Eq. (3.5), we obtain

$$\begin{aligned}
\langle x | K_n | y \rangle &= (-1)^n \frac{e^{-(x-y)^2/(4\tau)}}{(4\pi\tau)^{\nu/2}} \int_0^\tau d\tau_n \cdots \int_0^{\tau_3} d\tau_2 \int_0^{\tau_2} d\tau_1 \\
&\times \exp \left(\frac{(\tau - \tau_1) \tau_1}{\tau} \nabla_{R_1}^2 + \cdots + \frac{(\tau - \tau_{n-1}) \tau_{n-1}}{\tau} \nabla_{R_{n-1}}^2 + \frac{(\tau - \tau_n) \tau_n}{\tau} \nabla_{R_n}^2 \right. \\
&+ 2 \left[\frac{(\tau - \tau_2) \tau_1}{\tau} \nabla_{R_1} \nabla_{R_2} + \cdots + \frac{(\tau - \tau_{n-1}) \tau_1}{\tau} \nabla_{R_1} \nabla_{R_{n-1}} + \frac{(\tau - \tau_n) \tau_1}{\tau} \nabla_{R_1} \nabla_{R_n} \right] \\
&+ 2 \left[\frac{(\tau - \tau_3) \tau_2}{\tau} \nabla_{R_2} \nabla_{R_3} + \cdots + \frac{(\tau - \tau_{n-1}) \tau_2}{\tau} \nabla_{R_2} \nabla_{R_{n-1}} + \frac{(\tau - \tau_n) \tau_2}{\tau} \nabla_{R_2} \nabla_{R_n} \right] \\
&+ \cdots + 2 \frac{(\tau - \tau_n) \tau_{n-1}}{\tau} \nabla_{R_{n-1}} \nabla_{R_n} \Big) V(R_1) V(R_2) \cdots V(R_n). \tag{3.11}
\end{aligned}$$

4 An alternative expression of the n -th order heat kernels

In this section, we suggest an alternative expression of the n -th order matrix element of heat kernel $\langle x | K_n | y \rangle$, in which the integral with variable upper limit given in the above section is replaced by definite integrals.

The integral in Eq. (3.11) is a variable upper limit integral. In the following we rewrite Eq. (3.11) by a definite integral.

Introducing

$$\tau_1 = u_1 \tau_2, \quad \tau_2 = u_2 \tau_3, \quad \cdots \tau_{n-1} = u_{n-1} \tau_n, \quad \tau_n = u_n \tau, \tag{4.1}$$

we can rewrite Eq. (3.11) as

$$\begin{aligned}
\langle x | K_n | y \rangle &= (-1)^n \frac{e^{-(x-y)^2/(4\tau)}}{(4\pi\tau)^{\nu/2}} \tau^n \int_0^1 u_n^{n-1} du_n \int_0^1 u_{n-1}^{n-2} du_{n-1} \cdots \int_0^1 u_2 du_2 \int_0^1 du_1 \\
&\times \exp \left((1 - u_1 u_2 \cdots u_n) u_1 u_2 \cdots u_n \tau \nabla_{R_1}^2 + \cdots + (1 - u_{n-1} u_n) u_{n-1} u_n \tau \nabla_{R_{n-1}}^2 \right. \\
&+ (1 - u_n) u_n \tau \nabla_{R_n}^2 + 2 [(1 - u_2 \cdots u_n) u_1 u_2 \cdots u_n \tau \nabla_{R_1} \nabla_{R_2} + \cdots \\
&+ (1 - u_{n-1} u_n) u_1 u_2 \cdots u_n \tau \nabla_{R_1} \nabla_{R_{n-1}} + (1 - u_n) u_1 u_2 \cdots u_n \tau \nabla_{R_1} \nabla_{R_n}] \\
&+ 2 [(1 - u_3 \cdots u_n) u_2 \cdots u_n \tau \nabla_{R_2} \nabla_{R_3} + \cdots + (1 - u_{n-1} u_n) u_2 \cdots u_n \tau \nabla_{R_2} \nabla_{R_{n-1}} \\
&+ (1 - u_n) u_2 \cdots u_n \tau \nabla_{R_2} \nabla_{R_n}] + \cdots \\
&+ 2 [(1 - u_{n-1} u_n) u_{n-2} u_{n-1} u_n \tau \nabla_{R_{n-2}} \nabla_{R_{n-1}} + (1 - u_n) u_{n-2} u_{n-1} u_n \tau \nabla_{R_{n-2}} \nabla_{R_n}] \\
&+ 2 (1 - u_n) u_{n-1} u_n \tau \nabla_{R_{n-1}} \nabla_{R_n} \Big) V(R_1) \cdots V(R_{n-1}) V(R_n). \tag{4.2}
\end{aligned}$$

This expression of $\langle x | K_n | y \rangle$ also can be equivalently represented as

$$\begin{aligned} \langle x | K_n | y \rangle &= (-1)^n \frac{e^{-(x-y)^2/(4\tau)}}{(4\pi\tau)^{\nu/2}} \tau^n \int_0^1 u_n^{n-1} du_n \int_0^1 u_{n-1}^{n-2} du_{n-1} \cdots \int_0^1 u_2 du_2 \int_0^1 du_1 \\ &\times \exp \left(\left((1 - u_1 u_2 \cdots u_n) \nabla_{R_1}, (1 - u_2 \cdots u_n) \nabla_{R_2}, \cdots, (1 - u_{n-1} u_n) \nabla_{R_{n-1}}, (1 - u_n) \nabla_{R_n} \right) \right. \\ &\times \left. \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 2 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 & 2 & \cdots & 1 & 0 \\ 2 & 2 & \cdots & 2 & 1 \end{pmatrix} \begin{pmatrix} u_1 u_2 \cdots u_n \tau \nabla_{R_1} \\ u_2 \cdots u_n \tau \nabla_{R_2} \\ \vdots \\ u_{n-1} u_n \tau \nabla_{R_{n-1}} \\ u_n \tau \nabla_{R_n} \end{pmatrix} \right) V(R_1) \cdots V(R_{n-1}) V(R_n). \end{aligned} \quad (4.3)$$

5 The first three orders of diagonal and global heat kernels

In this section, based on the above results, we give two equivalent represents of the first three orders of diagonal and global heat kernels: one is represented by $\tilde{V}(k_n)$ and the other is represented by $V(R_n)$.

5.1 The heat kernel with respect to $\tilde{V}(k_n)$

First-order contribution. The first-order heat kernel $\langle x | K_1 | y \rangle$ in the coordinate representation, by Eq. (3.9), is

$$\begin{aligned} \langle x | K_1 | y \rangle &= -\frac{e^{-(x-y)^2/(4\tau)}}{(4\pi\tau)^{\nu/2}} \int_0^\tau d\tau_1 \int \frac{d^\nu k}{(2\pi)^{\nu/2}} \tilde{V}(k) \\ &\times \exp \left(ik \frac{\tau_1 x + (\tau - \tau_1) y}{\tau} - \frac{(\tau - \tau_1) \tau_1}{\tau} k^2 \right). \end{aligned} \quad (5.1)$$

The first-order diagonal heat kernel is then

$$\langle x | K_1 | x \rangle = -\frac{1}{(4\pi\tau)^{\nu/2}} \int_0^\tau d\tau_1 \int \frac{d^\nu k}{(2\pi)^{\nu/2}} \tilde{V}(k) \exp \left(ikx - \frac{(\tau - \tau_1) \tau_1}{\tau} k^2 \right). \quad (5.2)$$

The first-order global heat kernel then reads

$$\begin{aligned} K_1(\tau) &= \int \langle x | K_1 | x \rangle d^\nu x \\ &= -\frac{2\pi}{(4\pi\tau)^{\nu/2}} \int_0^\tau d\tau_1 \int \frac{d^\nu k}{(2\pi)^{\nu/2}} \tilde{V}(k) \exp \left(-\frac{(\tau - \tau_1) \tau_1}{\tau} k^2 \right) \delta(k) \\ &= -\frac{1}{2^{(3\nu-2)/2} \pi^{\nu-1} \tau^{(\nu-2)/2}} \tilde{V}(0). \end{aligned} \quad (5.3)$$

Second-order contribution. The second-order heat kernel $\langle x|K_2|y\rangle$ in the coordinate representation, by Eq. (3.9), is

$$\begin{aligned}\langle x|K_2|y\rangle &= \frac{e^{-(x-y)^2/4\tau}}{(4\pi\tau)^{\nu/2}} \int_0^\tau d\tau_2 \int \frac{d^\nu k_2}{(2\pi)^{\nu/2}} \tilde{V}(k_2) \int_0^{\tau_2} d\tau_1 \int \frac{d^\nu k_1}{(2\pi)^{\nu/2}} \tilde{V}(k_1) \\ &\quad \times \exp\left(ik_1\left(\frac{\tau-\tau_1}{\tau}y + \frac{\tau_1}{\tau}x\right) + ik_2\left(\frac{\tau-\tau_2}{\tau}y + \frac{\tau_2}{\tau}x\right) \right. \\ &\quad \left. - \frac{(\tau-\tau_1)\tau_1}{\tau}k_1^2 - \frac{(\tau-\tau_2)\tau_2}{\tau}k_2^2 - \frac{(\tau-\tau_2)\tau_1}{\tau}2k_2k_1\right).\end{aligned}\quad (5.4)$$

The second-order diagonal heat kernel is

$$\begin{aligned}\langle x|K_2|x\rangle &= \frac{1}{(4\pi\tau)^{\nu/2}} \int_0^\tau d\tau_2 \int \frac{d^\nu k_2}{(2\pi)^{\nu/2}} \tilde{V}(k_2) \int_0^{\tau_2} d\tau_1 \int \frac{d^\nu k_1}{(2\pi)^{\nu/2}} \tilde{V}(k_1) \\ &\quad \times \exp\left(ik_1x + ik_2x - \frac{(\tau-\tau_1)\tau_1}{\tau}k_1^2 - \frac{(\tau-\tau_2)\tau_2}{\tau}k_2^2 - \frac{(\tau-\tau_2)\tau_1}{\tau}2k_2k_1\right).\end{aligned}\quad (5.5)$$

The second-order global heat kernel then reads

$$\begin{aligned}K_2(\tau) &= \int \langle x|K_2|x\rangle d^\nu x \\ &= \frac{1}{(4\pi\tau)^{\nu/2}} \int_0^\tau d\tau_2 \int \frac{d^\nu k_2}{(2\pi)^{\nu/2}} \tilde{V}(k_2) \int_0^{\tau_2} d\tau_1 \int \frac{d^\nu k_1}{(2\pi)^{\nu/2}} \tilde{V}(k_1) \\ &\quad \times \exp\left(-\frac{(\tau-\tau_1)\tau_1}{\tau}k_1^2 - \frac{(\tau-\tau_2)\tau_2}{\tau}k_2^2 - \frac{(\tau-\tau_2)\tau_1}{\tau}2k_2k_1\right) \delta(k_1+k_2) \\ &= \frac{1}{2^{(3\nu+2)/2}\pi^{(2\nu-1)/2}\tau^{(\nu-3)/2}} \int \frac{d^\nu k_2}{(2\pi)^{\nu/2}} \frac{e^{-k_2^2\tau/4}}{k_2} \operatorname{erfi}\left(\frac{k_2\sqrt{\tau}}{2}\right) \tilde{V}^2(k_2),\end{aligned}\quad (5.6)$$

where $\operatorname{erfi}(z)$ gives the imaginary error function $\operatorname{erf}(z)/i$.

Third-order contribution. The third-order heat kernel $\langle x|K_3|y\rangle$ in the coordinate representation, by Eq. (3.9), is

$$\begin{aligned}\langle x|K_3|y\rangle &= -\frac{e^{-(x-y)^2/(4\tau)}}{(4\pi\tau)^{\nu/2}} \int_0^\tau d\tau_3 \int \frac{d^\nu k_3}{(2\pi)^{\nu/2}} \tilde{V}(k_3) \int_0^{\tau_3} d\tau_2 \int \frac{d^\nu k_2}{(2\pi)^{\nu/2}} \tilde{V}(k_2) \int_0^{\tau_2} d\tau_1 \int \frac{d^\nu k_1}{(2\pi)^{\nu/2}} \tilde{V}(k_1) \\ &\quad \times \exp\left(ik_1\left[\frac{\tau_1}{\tau}x + \frac{(\tau-\tau_1)}{\tau}y\right] + ik_2\left[\frac{\tau_2}{\tau}x + \frac{(\tau-\tau_2)}{\tau}y\right] + ik_3\left[\frac{\tau_3}{\tau}x + \frac{(\tau-\tau_3)}{\tau}y\right] \right. \\ &\quad \left. - \left[k_1^2\frac{(\tau-\tau_1)\tau_1}{\tau} + k_2^2\frac{(\tau-\tau_2)\tau_2}{\tau} + k_3^2\frac{(\tau-\tau_3)\tau_3}{\tau}\right] \right. \\ &\quad \left. - \left[2k_1k_2\frac{(\tau-\tau_2)\tau_1}{\tau} + 2k_1k_3\frac{(\tau-\tau_3)\tau_1}{\tau} + 2k_2k_3\frac{(\tau-\tau_3)\tau_2}{\tau}\right]\right).\end{aligned}\quad (5.7)$$

The third-order diagonal heat kernel $\langle x|K_3|x\rangle$ is

$$\begin{aligned}\langle x|K_3|x\rangle &= -\frac{1}{(4\pi\tau)^{\nu/2}} \int_0^\tau d\tau_3 \int \frac{d^\nu k_3}{(2\pi)^{\nu/2}} \tilde{V}(k_3) \int_0^{\tau_3} d\tau_2 \int \frac{d^\nu k_2}{(2\pi)^{\nu/2}} \tilde{V}(k_2) \int_0^{\tau_2} d\tau_1 \int \frac{d^\nu k_1}{(2\pi)^{\nu/2}} \tilde{V}(k_1) \\ &\quad \times \exp\left(ik_1x + ik_2x + ik_3x - \left[k_1^2\frac{(\tau-\tau_1)\tau_1}{\tau} + k_2^2\frac{(\tau-\tau_2)\tau_2}{\tau} + k_3^2\frac{(\tau-\tau_3)\tau_3}{\tau}\right] \right. \\ &\quad \left. - \left[2k_1k_2\frac{(\tau-\tau_2)\tau_1}{\tau} + 2k_1k_3\frac{(\tau-\tau_3)\tau_1}{\tau} + 2k_2k_3\frac{(\tau-\tau_3)\tau_2}{\tau}\right]\right).\end{aligned}\quad (5.8)$$

The third-order global heat kernel then reads

$$\begin{aligned}
K_3(\tau) &= \int \langle x | K_3 | x \rangle d^\nu x \\
&= -\frac{1}{(4\pi\tau)^{\nu/2}} \int_0^\tau d\tau_3 \int \frac{d^\nu k_3}{(2\pi)^{\nu/2}} \tilde{V}(k_3) \int_0^{\tau_3} d\tau_2 \int \frac{d^\nu k_2}{(2\pi)^{\nu/2}} \tilde{V}(k_2) \int_0^{\tau_2} d\tau_1 \int \frac{d^\nu k_1}{(2\pi)^{\nu/2}} \tilde{V}(k_1) \\
&\quad \times \exp \left(- \left[k_1^2 \frac{(\tau - \tau_1) \tau_1}{\tau} + k_2^2 \frac{(\tau - \tau_2) \tau_2}{\tau} + k_3^2 \frac{(\tau - \tau_3) \tau_3}{\tau} \right] \right. \\
&\quad \left. - \left[2k_1 k_2 \frac{(\tau - \tau_2) \tau_1}{\tau} + 2k_1 k_3 \frac{(\tau - \tau_3) \tau_1}{\tau} + 2k_2 k_3 \frac{(\tau - \tau_3) \tau_2}{\tau} \right] \right) \delta(k_1 + k_2 + k_3) \\
&= -\frac{1}{(4\pi\tau)^{\nu/2}} \int_0^\tau d\tau_3 \int \frac{d^\nu k_3}{(2\pi)^{\nu/2}} \tilde{V}(k_3) \int_0^{\tau_3} d\tau_2 \int \frac{d^\nu k_2}{(2\pi)^{\nu/2}} \tilde{V}(k_2) \int_0^{\tau_2} d\tau_1 \frac{1}{(2\pi)^{\nu/2}} \tilde{V}(-k_2 - k_3) \\
&\quad \times \exp \left(\frac{(\tau_1 - \tau_2)(\tau_1 - \tau_2 + \tau)}{\tau} k_2^2 + \frac{(\tau_1 - \tau_3)(\tau_1 - \tau_3 + \tau)}{\tau} k_3^2 + \frac{(\tau_1 - \tau_2)(\tau_1 - \tau_3 + \tau)}{\tau} 2k_2 k_3 \right). \tag{5.9}
\end{aligned}$$

5.2 The heat kernel with respect to $V(R_n)$

In this section we give an alternative expression of first three orders of diagonal and global heat kernels, which is represented $V(R_n)$, based on Eq. (3.11).

By Eq. (3.11), the first-order heat kernel $\langle x | K_1 | y \rangle$ is

$$\langle x | K_1 | y \rangle = -\frac{e^{-(x-y)^2/(4\tau)}}{(4\pi\tau)^{\nu/2}} \int_0^\tau d\tau_1 \exp \left(\frac{(\tau - \tau_1) \tau_1}{\tau} \nabla_{R_1}^2 \right) V(R_1). \tag{5.10}$$

The first-order diagonal heat kernel is

$$\langle x | K_1 | x \rangle = -\frac{1}{(4\pi\tau)^{\nu/2}} \int_0^\tau d\tau_1 \exp \left(\frac{(\tau - \tau_1) \tau_1}{\tau} \nabla_x^2 \right) V(x). \tag{5.11}$$

The first-order global heat kernel is

$$\begin{aligned}
K_1(\tau) &= \int d^\nu x \langle x | K_1 | x \rangle \\
&= -\frac{1}{(4\pi\tau)^{\nu/2}} \int_0^\tau d\tau_1 \int d^\nu x \exp \left(\frac{(\tau - \tau_1) \tau_1}{\tau} \nabla_x^2 \right) V(x). \tag{5.12}
\end{aligned}$$

The second-order heat kernel is

$$\begin{aligned}
\langle x | K_2 | y \rangle &= \frac{e^{-(x-y)^2/(4\tau)}}{(4\pi\tau)^{\nu/2}} \int_0^\tau d\tau_2 \int_0^{\tau_2} d\tau_1 \exp \left(\frac{(\tau - \tau_1) \tau_1}{\tau} \nabla_{R_1}^2 \right. \\
&\quad \left. + \frac{(\tau - \tau_2) \tau_2}{\tau} \nabla_{R_2}^2 + 2 \frac{(\tau - \tau_2) \tau_1}{\tau} \nabla_{R_1} \nabla_{R_2} \right) V(R_2) V(R_1). \tag{5.13}
\end{aligned}$$

The second-order diagonal heat kernel is

$$\begin{aligned}
\langle x | K_2 | x \rangle &= \frac{1}{(4\pi\tau)^{\nu/2}} \int_0^\tau d\tau_2 \int_0^{\tau_2} d\tau_1 \exp \left(\frac{(\tau - \tau_1) \tau_1}{\tau} \nabla_x^2 \right. \\
&\quad \left. + \frac{(\tau - \tau_2) \tau_2}{\tau} \nabla_x^2 + 2 \frac{(\tau - \tau_2) \tau_1}{\tau} \nabla_x^2 \right) V^2(x) \\
&= \frac{1}{(4\pi\tau)^{\nu/2}} \int_0^\tau d\tau_2 \int_0^{\tau_2} d\tau_1 \exp \left(\frac{\tau (3\tau_1 + \tau_2) - (\tau_1 + \tau_2)^2}{\tau} \nabla_x^2 \right) V^2(x). \quad (5.14)
\end{aligned}$$

The second-order global heat kernel is

$$\begin{aligned}
K_2(\tau) &= \int d^\nu x \langle x | K_2 | x \rangle \\
&= \frac{1}{(4\pi\tau)^{\nu/2}} \int_0^\tau d\tau_2 \int_0^{\tau_2} d\tau_1 \int d^\nu x \exp \left(\frac{(\tau - \tau_1) \tau_1}{\tau} \nabla_x^2 \right. \\
&\quad \left. + \frac{(\tau - \tau_2) \tau_2}{\tau} \nabla_x^2 + 2 \frac{(\tau - \tau_2) \tau_1}{\tau} \nabla_x^2 \right) V^2(x) \\
&= \frac{1}{(4\pi\tau)^{\nu/2}} \int_0^\tau d\tau_2 \int_0^{\tau_2} d\tau_1 \int d^\nu x \exp \left(\frac{\tau (3\tau_1 + \tau_2) - (\tau_1 + \tau_2)^2}{\tau} \nabla_x^2 \right) V^2(x). \quad (5.15)
\end{aligned}$$

The third-order heat kernel $\langle x | K_3 | y \rangle$ is

$$\begin{aligned}
\langle x | K_3 | y \rangle &= -\frac{e^{-(x-y)^2/(4\tau)}}{(4\pi\tau)^{\nu/2}} \int_0^\tau d\tau_3 \int_0^{\tau_3} d\tau_2 \int_0^{\tau_2} d\tau_1 \\
&\quad \times \exp \left(\frac{(\tau - \tau_1) \tau_1}{\tau} \nabla_{R_1}^2 + \frac{(\tau - \tau_2) \tau_2}{\tau} \nabla_{R_2}^2 + \frac{(\tau - \tau_3) \tau_3}{\tau} \nabla_{R_3}^2 + 2 \frac{(\tau - \tau_2) \tau_1}{\tau} \nabla_{R_1} \nabla_{R_2} \right. \\
&\quad \left. + 2 \frac{(\tau - \tau_3) \tau_1}{\tau} \nabla_{R_1} \nabla_{R_3} + 2 \frac{(\tau - \tau_3) \tau_2}{\tau} \nabla_{R_2} \nabla_{R_3} \right) V(R_1) V(R_2) V(R_3). \quad (5.16)
\end{aligned}$$

The third-order diagonal heat kernel is

$$\begin{aligned}
\langle x | K_3 | x \rangle &= -\frac{1}{(4\pi\tau)^{\nu/2}} \int_0^\tau d\tau_3 \int_0^{\tau_3} d\tau_2 \int_0^{\tau_2} d\tau_1 \\
&\quad \times \exp \left(\frac{(\tau - \tau_1) \tau_1}{\tau} \nabla_x^2 + \frac{(\tau - \tau_2) \tau_2}{\tau} \nabla_x^2 + \frac{(\tau - \tau_3) \tau_3}{\tau} \nabla_x^2 + 2 \frac{(\tau - \tau_2) \tau_1}{\tau} \nabla_x^2 \right. \\
&\quad \left. + 2 \frac{(\tau - \tau_3) \tau_1}{\tau} \nabla_x^2 + 2 \frac{(\tau - \tau_3) \tau_2}{\tau} \nabla_x^2 \right) V^3(x) \\
&= -\frac{1}{(4\pi\tau)^{\nu/2}} \int_0^\tau d\tau_3 \int_0^{\tau_3} d\tau_2 \int_0^{\tau_2} d\tau_1 \\
&\quad \times \exp \left(\frac{\tau (5\tau_1 + 3\tau_2 + \tau_3) - (\tau_1 + \tau_2 + \tau_3)^2}{\tau} \nabla_x^2 \right) V^3(x). \quad (5.17)
\end{aligned}$$

The third-order global heat kernel is

$$\begin{aligned}
K_3(\tau) &= \int \langle x | K_3 | x \rangle d^\nu x \\
&= -\frac{1}{(4\pi\tau)^{\nu/2}} \int_0^\tau d\tau_3 \int_0^{\tau_3} d\tau_2 \int_0^{\tau_2} d\tau_1 \int d^\nu x \\
&\quad \times \exp \left(\frac{(\tau - \tau_1)\tau_1}{\tau} \nabla_x^2 + \frac{(\tau - \tau_2)\tau_2}{\tau} \nabla_x^2 + \frac{(\tau - \tau_3)\tau_3}{\tau} \nabla_x^2 + 2\frac{(\tau - \tau_2)\tau_1}{\tau} \nabla_x^2 \right. \\
&\quad \left. + 2\frac{(\tau - \tau_3)\tau_1}{\tau} \nabla_x^2 + 2\frac{(\tau - \tau_3)\tau_2}{\tau} \nabla_x^2 \right) V^3(x). \\
&= -\frac{1}{(4\pi\tau)^{\nu/2}} \int_0^\tau d\tau_3 \int_0^{\tau_3} d\tau_2 \int_0^{\tau_2} d\tau_1 \int d^\nu x \\
&\quad \times \exp \left(\frac{\tau(5\tau_1 + 3\tau_2 + \tau_3) - (\tau_1 + \tau_2 + \tau_3)^2}{\tau} \nabla_x^2 \right) V^3(x). \tag{5.18}
\end{aligned}$$

5.3 The heat kernel with respect to u_n

In this section, we give an equivalent expression of the first three orders of diagonal and global heat kernels, which is represented u_n , based on Eq. (4.2).

By Eq. (4.2), the first-order heat kernel $\langle x | K_1 | y \rangle$ is

$$\langle x | K_1 | y \rangle = -\frac{e^{-(x-y)^2/(4\tau)}}{(4\pi\tau)^{\nu/2}} \tau \int_0^1 du_1 \exp((1-u_1)u_1\tau\nabla_{R_1}^2) V(R_1). \tag{5.19}$$

The first-order diagonal heat kernel is

$$\langle x | K_1 | x \rangle = -\frac{1}{(4\pi\tau)^{\nu/2}} \tau \int_0^1 du_1 \exp((1-u_1)u_1\tau\nabla_x^2) V(x). \tag{5.20}$$

The first-order global heat kernel is

$$\begin{aligned}
K_1(\tau) &= \int \langle x | K_1 | x \rangle d^\nu x \\
&= -\frac{1}{(4\pi\tau)^{\nu/2}} \tau \int_0^1 du_1 \int d^\nu x \exp((1-u_1)u_1\tau\nabla_x^2) V(x). \tag{5.21}
\end{aligned}$$

The second-order heat kernel $\langle x | K_2 | y \rangle$ is

$$\begin{aligned}
\langle x | K_2 | y \rangle &= \frac{e^{-(x-y)^2/(4\tau)}}{(4\pi\tau)^{\nu/2}} \tau^2 \int_0^1 u_2 du_2 \int_0^1 du_1 \exp((1-u_1u_2)u_1u_2\tau\nabla_{R_1}^2 \\
&\quad + (1-u_2)u_2\tau\nabla_{R_2}^2 + 2(1-u_2)u_1u_2\tau\nabla_{R_1}\nabla_{R_2}) V(R_2) V(R_1). \tag{5.22}
\end{aligned}$$

The second-order diagonal heat kernel is

$$\begin{aligned}
\langle x | K_2 | x \rangle &= \frac{1}{(4\pi\tau)^{\nu/2}} \tau^2 \int_0^1 u_2 du_2 \int_0^1 du_1 \exp((1-u_1u_2)u_1u_2\tau\nabla_x^2 \\
&\quad + (1-u_2)u_2\tau\nabla_x^2 + 2(1-u_2)u_1u_2\tau\nabla_x^2) V^2(x). \tag{5.23}
\end{aligned}$$

The second-order global heat kernel is

$$\begin{aligned}
K_2(\tau) &= \int \langle x | K_2 | x \rangle d^\nu x \\
&= \frac{1}{(4\pi\tau)^{\nu/2}} \tau^2 \int_0^1 u_2 du_2 \int_0^1 du_1 \int d^\nu x \exp \left((1 - u_1 u_2) u_1 u_2 \tau \nabla_x^2 \right. \\
&\quad \left. + (1 - u_2) u_2 \tau \nabla_x^2 + 2(1 - u_2) u_1 u_2 \tau \nabla_x^2 \right) V^2(x).
\end{aligned} \tag{5.24}$$

The third-order heat kernel $\langle x | K_3 | y \rangle$ is

$$\begin{aligned}
\langle x | K_3 | y \rangle &= -\frac{e^{-(x-y)^2/(4\tau)}}{(4\pi\tau)^{\nu/2}} \tau^3 \int_0^1 u_3^2 du_3 \int_0^1 u_2 du_2 \int_0^1 du_1 \\
&\quad \times \exp \left((1 - u_1 u_2 u_3) u_1 u_2 u_3 \tau \nabla_{R_1}^2 + (1 - u_2 u_3) u_2 u_3 \tau \nabla_{R_2}^2 \right. \\
&\quad \left. + (1 - u_3) u_3 \tau \nabla_{R_3}^2 + 2(1 - u_2 u_3) u_1 u_2 u_3 \tau \nabla_{R_1} \nabla_{R_2} \right. \\
&\quad \left. + 2(1 - u_3) u_1 u_2 u_3 \tau \nabla_{R_1} \nabla_{R_3} + 2(1 - u_3) u_2 u_3 \tau \nabla_{R_2} \nabla_{R_3} \right) V(R_1) V(R_2) V(R_3).
\end{aligned} \tag{5.25}$$

The third-order diagonal heat kernel is

$$\begin{aligned}
\langle x | K_3 | x \rangle &= -\frac{1}{(4\pi\tau)^{\nu/2}} \tau^3 \int_0^1 u_3^2 du_3 \int_0^1 u_2 du_2 \int_0^1 du_1 \\
&\quad \times \exp \left((1 - u_1 u_2 u_3) u_1 u_2 u_3 \tau \nabla_x^2 + (1 - u_2 u_3) u_2 u_3 \tau \nabla_x^2 \right. \\
&\quad \left. + (1 - u_3) u_3 \tau \nabla_x^2 + 2(1 - u_2 u_3) u_1 u_2 u_3 \tau \nabla_x^2 \right. \\
&\quad \left. + 2(1 - u_3) u_1 u_2 u_3 \tau \nabla_x^2 + 2(1 - u_3) u_2 u_3 \tau \nabla_x^2 \right) V^3(x).
\end{aligned} \tag{5.26}$$

The third-order global heat kernel is

$$\begin{aligned}
K_3(\tau) &= \int \langle x | K_3 | x \rangle d^\nu x \\
&= -\frac{1}{(4\pi\tau)^{\nu/2}} \tau^3 \int_0^1 u_3^2 du_3 \int_0^1 u_2 du_2 \int_0^1 du_1 \int d^\nu x \\
&\quad \times \exp \left((1 - u_1 u_2 u_3) u_1 u_2 u_3 \tau \nabla_x^2 + (1 - u_2 u_3) u_2 u_3 \tau \nabla_x^2 \right. \\
&\quad \left. + (1 - u_3) u_3 \tau \nabla_x^2 + 2(1 - u_2 u_3) u_1 u_2 u_3 \tau \nabla_x^2 \right. \\
&\quad \left. + 2(1 - u_3) u_1 u_2 u_3 \tau \nabla_x^2 + 2(1 - u_3) u_2 u_3 \tau \nabla_x^2 \right) V^3(x).
\end{aligned} \tag{5.27}$$

6 Conclusion

The heat-kernel method is widely used in physics and mathematics. In application, there are few heat kernel which can be solved exactly. Thus in most cases, we need to solve the heat kernel approximately. As a kind of approximation methods, there are many asymptotic method for solving heat kernels, such as the Seeley-DeWitt type expansion [1] and the covariant perturbation expansion [19–29].

In literature, the covariant perturbation expansion is only used to calculate diagonal heat kernels. In this paper, we use the covariant perturbation expansion to calculate non-diagonal heat kernel.

The non-diagonal heat kernel contains all information of an operator. Therefore, starting from a non-diagonal heat kernel, we can calculate many physical quantities, such as effective actions, vacuum energies [4], spectral counting functions [8], thermodynamic quantities, and scattering phase shifts [9, 10]. In the further work, we will calculate such quantities by the result given by the present paper.

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References

- [1] D. V. Vassilevich, *Heat kernel expansion: user's manual*, *Physics Reports* **388** (2003), no. 5 279–360.
- [2] K. Kirsten, *Spectral functions in mathematics and physics*. CRC Press, 2001.
- [3] E. B. Davies, *Heat kernels and spectral theory*, vol. 92. Cambridge University Press, 1990.
- [4] W.-S. Dai and M. Xie, *An approach for the calculation of one-loop effective actions, vacuum energies, and spectral counting functions*, *Journal of High Energy Physics* **2010** (2010), no. 6 1–29.
- [5] V. Mukhanov and S. Winitzki, *Introduction to quantum effects in gravity*. Cambridge University Press, 2007.
- [6] W.-S. Dai and M. Xie, *Quantum statistics of ideal gases in confined space*, *Physics Letters A* **311** (2003), no. 4 340–346.
- [7] W.-S. Dai and M. Xie, *Geometry effects in confined space*, *Physical Review E* **70** (2004), no. 1 016103.
- [8] W.-S. Dai and M. Xie, *The number of eigenstates: counting function and heat kernel*, *Journal of High Energy Physics* **2009** (2009), no. 02 033.
- [9] W.-D. Li and W.-S. Dai, *Heat-kernel approach for scattering*, *The European Physical Journal C* **75** (2015), no. 6.
- [10] H. Pang, W.-S. Dai, and M. Xie, *Relation between heat kernel method and scattering spectral method*, *The European Physical Journal C-Particles and Fields* **72** (2012), no. 5 1–13.
- [11] D. Fliegner, M. G. Schmidt, and C. Schubert, *The higher derivative expansion of the effective action by the string-inspired method. i*, *Zeitschrift für Physik C Particles and Fields* **64** (1994), no. 1 111–116.
- [12] D. Fliegner, P. Haberl, M. Schmidt, and C. Schubert, *The higher derivative expansion of the effective action by the string inspired method, ii*, *Annals of Physics* **264** (1998), no. 1 51–74.
- [13] R. I. Nepomechie, *Calculating heat kernels*, *Physical Review D* **31** (1985), no. 12 3291.
- [14] L. Culumovic and D. McKeon, *Calculation of off-diagonal elements of the heat kernel*, *Physical Review D* **38** (1988), no. 12 3831.
- [15] F. Dilkes and D. McKeon, *Off-diagonal elements of the dewitt expansion from the quantum-mechanical path integral*, *Physical Review D* **53** (1996), no. 8 4388.

- [16] D. McKEON, *Seeley-gilkey coefficients for superoperators*, *Modern Physics Letters A* **6** (1991), no. 40 3711–3715.
- [17] M. Kotani and T. Sunada, *Albanese maps and off diagonal long time asymptotics for the heat kernel*, *Communications in Mathematical Physics* **209** (2000), no. 3 633–670.
- [18] K. Groh, F. Saueressig, and O. Zanusso, *Off-diagonal heat-kernel expansion and its application to fields with differential constraints*, *arXiv preprint arXiv:1112.4856* (2011).
- [19] A. Barvinsky and G. Vilkovisky, *The generalized schwinger-dewitt technique in gauge theories and quantum gravity*, *Physics Reports* **119** (1985), no. 1 1–74.
- [20] A. Barvinsky and G. Vilkovisky, *Beyond the schwinger-dewitt technique: Converting loops into trees and in-in currents*, *Nuclear Physics B* **282** (1987) 163–188.
- [21] A. Barvinsky, Y. V. Gusev, G. Vilkovisky, and V. Zhytnikov, *The one-loop effective action and trace anomaly in four dimensions*, *Nuclear Physics B* **439** (1995), no. 3 561–582.
- [22] G. M. Shore, *A local effective action for photon-gravity interactions*, *Nuclear Physics B* **646** (2002), no. 1 281–300.
- [23] A. Barvinsky, V. Zhytnikov, Y. V. Gusev, and G. Vilkovisky, *Covariant perturbation theory; 4, third order in the curvature*, tech. rep., P00011539, 1993.
- [24] A. Barvinsky, Y. V. Gusev, G. Vilkovisky, and V. Zhytnikov, *Asymptotic behaviors of the heat kernel in covariant perturbation theory*, *Journal of Mathematical Physics* **35** (1994), no. 7 3543–3559.
- [25] A. Barvinsky and G. Vilkovisky, *Covariant perturbation theory (ii). second order in the curvature. general algorithms*, *Nuclear Physics B* **333** (1990), no. 2 471–511.
- [26] A. Barvinsky and G. Vilkovisky, *Covariant perturbation theory (iii). spectral representations of the third-order form factors*, *Nuclear Physics B* **333** (1990), no. 2 512–524.
- [27] Y. V. Gusev, *Heat kernel expansion in the covariant perturbation theory*, *Nuclear physics B* **807** (2009), no. 3 566–590.
- [28] A. Barvinsky and V. Mukhanov, *New nonlocal effective action*, *Physical Review D* **66** (2002), no. 6 065007.
- [29] A. Barvinsky and D. Nesterov, *Schwinger-dewitt technique for quantum effective action in brane induced gravity models*, *Physical Review D* **81** (2010), no. 8 085018.
- [30] A. E. Van de Ven, *Index-free heat kernel coefficients*, *Classical and Quantum Gravity* **15** (1998), no. 8 2311.